Lecture 3

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Earlier: An $O(n \log n)$ algo for PROP cake division
EF for 3 agents
Pareto optimality, & a threshold condn. for 2 agents.

Fractional knapsack:

Given a knapsack of size 1, n items, each has value $v_i > 0$, size $s_i > 0$,
I want to maximize value of items in my knapsack, and am allowed to pick items fractionally, i.e.,

$$\max \sum_i v_i x_i \qquad (x_i \in [0,1] \text{ is fraction of item picked})$$
$$\text{s.t.} \sum_i s_i x_i \leq 1$$
$$\forall i \quad x_i \in [0,1]$$

Algo:

Sort items so that $\frac{v_1}{s_1} \geq \frac{v_2}{s_2} \geq \cdots \geq \frac{v_n}{s_n}$
$S \leftarrow 1$ (remaining size of knapsack)
for $i = 1 \ldots n$
  if $s_i \leq S$ then
    $x_i = 1$, $S \leftarrow S - s_i$
  else $x_i = S/s_i$, break

Claim: Above algo gives an optimal soln. to fractional knapsack (prove yourself)

Claim: For any optimal soln x, $\exists$ a threshold θ s.t. $v_i/s_i > \theta \Rightarrow i \in$ optimal set ($x_i = 1$)
$v_i/s_i < \theta \Rightarrow i \notin$ optimal set ($x_i = 0$)

Proof: If v_i/s_i are equal $\forall i$, let $\theta = v_i/s_i$, this satisfies the claim.
Else, $\exists i, i'$ s.t. $v_i/s_i > v_{i'}/s_{i'}$

Recall: items sorted so that $v_i/s_i \geq v_{i+1}/s_{i+1}$, & $v_i > 0$, $s_i > 0$ $\forall i$.

Let $k = \arg\max \{i : x_i > 0\}$, $\theta = v_k/s_k$.

Suppose $\exists i : v_i/s_i > v_k/s_k$, $x_i < 1$
Choose $\delta > 0$ s.t. $x_i + \delta \leq 1$, $s_i\delta \leq s_k x_k$
Then suppose instead of $\frac{s_i}{s_k}\delta$ fraction of x_k, we take δ fraction of x_i
Change in value is $v_i\delta - v_k \frac{s_i}{s_k}\delta = \frac{\delta}{s_i}\left(\frac{v_i}{s_i} - \frac{v_k}{s_k}\right) > 0$

Similarly, suppose $\exists i : v_i/s_i < v_k/s_k$ & $x_k > 0$
Can give a similar argument, to show a contradiction ∎

This threshold is v_i/s_i of last item packed (possibly fractionally) in the knapsack.

There is also a nice dual interpretation:

$$\begin{array}{llll|llll} & \max & \sum_i v_i x_i & & & \min & \lambda + \sum_i p_i & \\ \text{s.t.} & \sum_i s_i x_i & \leq 1 & \times\lambda & \text{s.t.} \ \forall i & \lambda s_i + p_i & \geq v_i & \times x_i \\ \forall i & x_i & \leq 1 & \times p_i & & \lambda, p_i & \geq 0 & \\ \forall i & x_i & \geq 0 & & & & & \end{array}$$

A primal-dual interpretation: Say x^*, λ^*, p^* are optimal primal & dual variables

Then $\lambda^* = 0 \Rightarrow \forall i \ p_i^* \geq v_i^* \Rightarrow$ optimal value $= \sum_i v_i^*$
$\Rightarrow$ all items selected.
Else $\lambda^* > 0 \Rightarrow \sum_i s_i x_i^* = 1 \Rightarrow$ knapsack is full.
Now $x_i^* < 1 \Rightarrow p_i^* = 0 \Rightarrow v_i/s_i \leq \lambda^*$
$x_i^* = 1 \Rightarrow \lambda^* s_i + p_i^* = v_i \Rightarrow v_i/s_i \geq \lambda^*$

A market for goods.

- each agent has a valuation density fn. $f_i : C \to \mathbb{R}_+$
- there is a price density fn. $p : C \to \mathbb{R}_+$, so the cost of any piece is $\int_S p(x)\,dx$
- each agent has 1 Re to spend.

This is called a **Fisher market**.

If you were an agent, what cake would you "buy" to maximize their value? You have no utility for any left-over money.

Ans: $S \subseteq [0,1] : p(S) \leq 1$ that maximizes $v_i(S)$
let any such set S be called the demand set, i.e.
demand set of $i = \{ S \subseteq C : p(S) \leq 1$, and S maximizes $v_i(S)$ subject to this $\}$

Claim: $\forall i \ \exists$ threshold α_i s.t., for an optimal subset S, upto measure 0 sets,
$\{x : f_i(x)/p_i(x) > \alpha_i\} \subseteq S$
$\{x : f_i(x)/p_i(x) < \alpha_i\} \cap S = \phi$
(prove yourself – use the earlier exchange argument)

Defn.: A price density fn. $p : C \to \mathbb{R}_+$ and an allocation $A = (A_1, \ldots, A_n)$ of the cake is called a "Competitive Equilibrium w/ Equal Incomes" if:

1. A partitions the cake,
2. Each A_i is in it's demand set, given prices p,
3. $p(A_i) = 1$

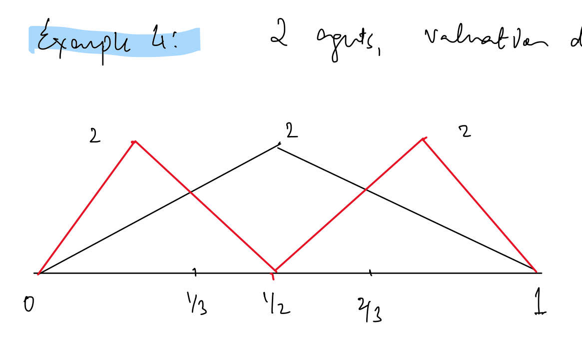
Note that from 1 & 3, $p(C) = n$

Example 1:
2 agents, $f_1(x) = f_2(x) = 1$
CEEI: $p(x) = 2$, $A_1 = [0, 1/2]$, $A_2 = [1/2, 1]$
Are there other equilibria?

Example 2:
2 agents, $f_1(x) = 2x$, $f_2(x) = 1$
CEEI: $p(x) = 2$, $A_1 = [1/2, 1]$, $A_2 = [0, 1]$

Example 3:
2 agents, $f_1(x) = f_2(x) = 2x$. Now if we set $p(x) = 2$, can't get an equilibrium.
Say $p(x) = 4x$, $A_1 = [0, 1/\sqrt{2}]$, $A_2 = [1/\sqrt{2}, 1]$
(how would you check this is an equilibrium?)

Example 4: 2 agents, valuation densities as shown.



Find a CEEI yourself.

Lemma: A CEEI allocation is EF.
(since if $v_i(A_i) < v_i(A_j)$, i could have just bought A_j)

For the next lemma, we need non-satiation: $\forall i, x \in C$, $f_i(x) > 0$
(i.e., agents are not satiated / satisfied w/ a smaller piece of cake.

Lemma: If agents are not satiable, then a CEEI allocation is PO.

Proof: Let (p, A) be a CEEI, & let B Pareto-dominate A.
Consider i: $v_i(B_i) \geq v_i(A_i)$. Then $p(A_i) = 1$ (since A is a CEEI).

Claim: $p(B_i) \geq 1$

Proof: Suppose $p(B_i) < 1$. Since $v_i(A_i) = 1$, $\exists$ set $S \subseteq A_i \setminus B$ s.t. $v_i(S) > 0$, $p(S) + p(B_i) \leq 1$. But then $v_i(S \cup B_i) > v_i(B_i) \geq v_i(A_i)$, & (by non-satiation) A_i could not have been in i's demand set.
Consider i: $v_i(B_i) > v_i(A_i)$. Then $p(B_i) > 1$, else A_i could not have been in i's demand set.
Then $p(C) = \sum_{i=1}^n p(B_i) > n$, giving a contradiction. ∎

Thus for non-satiable valuations a CEEI allocation is EF + PO.

Q. Give an example where non-satiability does not hold, and a CEEI allocation is not EF + PO.

A Welfare Objective

Consider the following welfare objectives: choose allocation $A = (A_1, \ldots, A_n)$ to maximize

1. $\frac{1}{n}\sum_i v_i(A_i)$ (the utilitarian social welfare)
2. $\min_i v_i(A_i)$ (the egalitarian social welfare)
3. $\left(\prod_i v_i(A_i)\right)^{1/n}$ (the Nash welfare)

Of these, only the Nash welfare is scale invariant.

This problem: $\max \left(\prod_i v_i(A_i)\right)^{1/n}$, s.t. A is a partition of the cake, is called the Eisenberg-Gale convex program.

Theorem: $\exists$ an allocation that maximizes the Nash welfare, any any allocation that maximizes the NW is a CEEI.
Conversely, any CEEI where every agent is non-satiated (i.e., $v_i(A_i) < 1$) maximizes NW.

We will prove this for simple valuation classes later.